1. Let $S = (a, b]$, where $a < b$. Determine the supremum and infimum of S.

2. Let $S = \{ \frac{n}{2^n} : n \in \mathbb{N} \}$. Show that $\sup S = \frac{1}{2}$. Think about what $\inf S$ is?
(Hints: binomial theorem)

3. Let A and B be bounded nonempty subset of $\mathbb{R}$, and let $A+B := \{ a+b : a \in A, b \in B \}$. Prove that

$$\sup (A+B) = \sup A + \sup B, \text{ and}$$
$$\inf (A+B) = \inf A + \inf B$$

4. Let X and Y be non-empty sets and let $h : X \times Y \to \mathbb{R}$ have bounded range in $\mathbb{R}$.
Let $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ be defined by

$$f(x) = \sup \{ h(x,y) : y \in Y \}, \quad g(y) = \inf \{ h(x,y) : x \in X \}$$

Prove that

$$\sup \{ g(y) : y \in Y \} \leq \inf \{ f(x) : x \in X \}$$

1. $\sup S = b$, $\inf S = a$ (supremum or infimum may be not belonged to the set S)

Let $u = \sup S$,

u is upper bound of S,

$u \geq b \in S$,

Also, since $b \geq x \quad \forall x \in S$, b is upper bound

Then by definition of supremum, $u \leq b$

$\Rightarrow \quad u = b$

S is non empty

Definition of sup.

If S is bounded above, then u is said to be a supremum (infimum) of S if

(1) u is upper (lower) bound of S

(2) if v is any (lower) upper bound of S, $u \leq v$ $(u \geq v)$

2. ~~Use~~ Apply binomial theorem to 2^n

$$2^n = (1+1)^n = \binom{n}{0} + \underbrace{\binom{n}{1}}_{=n} + \dots + \underbrace{\binom{n}{n-1}}_{=n} + \binom{n}{n}$$

$$\geq 2n$$

Then, we have $\frac{n}{2^n} \leq \frac{n}{2n} = \frac{1}{2} \Rightarrow \frac{1}{2}$ is upper bound of S

Let $u = \sup S$

By definition of supremum, $u \leq \frac{1}{2}$

Also, for $n=1$, $u \geq \frac{1}{2^1} = \frac{1}{2}$

Then, $u = \frac{1}{2}$

3. By definition, $\sup(A+B) \geq a+b \quad \forall a \in A, b \in B$

fix b, $\sup(A+B) - b \geq a \quad \forall a \in A$

Then $\sup(A+B) - b$ is upper bound of A

$\sup(A+B) - b \geq \sup A \quad \forall b \in B$

$\sup(A+B) - \sup A \geq b \quad \forall b \in B$

$\sup(A+B) - \sup A$ is upper bound of B

$\sup(A+B) - \sup A \geq \sup B$

$\Rightarrow \quad \sup(A+B) \geq \sup A + \sup B$

3 Then we need to show $\sup(A+B) \le \sup A + \sup B$

$$\sup A \ge a, \quad \sup B \ge b \quad \forall a, b$$

$$\sup A + \sup B \ge a+b \quad \forall a \in A, b \in B$$

$\sup A + \sup B$ is upper bound of $A+B$

$$\sup A + \sup B \ge \sup(A+B)$$

To conclude, $\sup A + \sup B = \sup(A+B)$

4. Since h has bounded range in $\mathbb{R}$

$\exists M \in \mathbb{R}$ s.t. $|h(x,y)| \le M \quad \forall x \in X, y \in Y$

Then $\{h(x,y) : y \in Y\}$ is bounded above

$\{h(x,y) : x \in X\}$ is bounded below.

Hence $f(x), g(y)$ is well defined.

Let $S_f = \{f(x) : x \in X\}$, $S_g = \{g(y) : y \in Y\}$

Then we need to show $\sup S_g \le \inf S_f$

if t_1 is true $\sup S_g \le \inf S_f$ by definition of inf.

t_1: $\sup S_g$ is lower bound of S_f
i.e. $\sup S_g \le f(x) \quad \forall x \in X$

Pf of t_1: Since $f(x) = \sup\{h(x,y) : y \in Y\}$

$$f(x) \ge h(x,y) \quad \forall y \in Y$$

Since $g(y) = \inf\{h(x,y) : x \in X\}$

$$h(x,y) \ge g(y) \quad \forall x \in X$$

Then $f(x) \ge \underline{g(y) \quad \forall y \in Y}$ S_g

Then $f(x)$ is upper bound of S_g

$$f(x) \ge \sup S_g$$

2. $\inf S = 0$

Since $\frac{n}{2^n} \geq 0$, $\inf S \geq 0$

it only need to show $\inf S \leq 0$

By MI, $2^n \geq n^2$, for $n \geq 4$

$\inf S \leq \frac{n}{2^n} \leq \frac{n}{n^2} \leq \frac{1}{n} < \varepsilon$, By Archimedean Property $\forall \varepsilon > 0$ $\exists n \in \mathbb{N}$ s.t. $\frac{1}{\varepsilon} < n$

~~Since~~ then $\inf S < \varepsilon$

since $\varepsilon > 0$ is arbitrary, Thm 2.1.9 $\Rightarrow \inf S = 0$

If $a \in \mathbb{R}$, $0 \leq a < \varepsilon$ $\forall \varepsilon > 0$

then $a = 0$